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IMTech 5*b*

Newsletter

Issue 5, Advances 5*b*, April 14, 2023

PhD highlights ◇ Alberto Larrauri

Outreach ◇ Luis Ángel Caffarelli

Chronicles ◇ Juan Luis Vázquez

Interviews ◇ Jaume Franch

Reviews

◇ Joaquín Pérez: The Gaussian Double-Bubble Conjectures

◇ **AMS:** Fully Nonlinear Elliptic Equations (1995, authors L. A. Caffarelli & X. Cabré)

Events

◇ Eva Miranda: Hardy tour

◇ Ingrid Daubechies: Wolf Prize 2023

◇ Albert Atserias: ICREA Academia

◇ Jordi Guàrdia: FME's New Dean

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IMTech Newsletter

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нет войне – No War!

Contents

❑ Editorial, 1
❑ Research, 8
❑ Outreach, 11
❑ Reviews, 20
❑ Contacts, 24
❑ Interviews, 2
❑ PhDs, 10
❑ Chronicles, 14
❑ Events, 23
❑ Idx NL05, 25

Editorial

As reported in NL05a, NL05b is the second Advance of NL05, released on April 14, 2023. It features the following items:

- ❑ An interiew with Jaume Franch, on the occasion of ending his second mandate as dean of the **FME**.
- ❑ The PhD highlight of Alberto Larrauri
- ❑ In a first celebration of the Abel Prize 2023 to Luis Ángel Caffarelli, we include the following items:
  - An Outeach article by Luis A. Caffarelli, which is an English translation of the inaugural lecture he delivered at the **FME** for the academic term 2003-2004.
  - A "personal glimpse" on Luis A. Caffarelli by Juan Luis Vázquez
  - A reproduction of the John Urbas' review of the landmark book *Fully nonlinear elliptic equations* published in 1995 by Luis Caffarelli and Xavier Cabré, with an annex quoting other reviews and providing indications of the sustained impact of the text.
- ❑ A review by Joaquín Pérez of the paper The Gaussian Double-Bubble and Multi-Bubble Conjectures recently published in the *Annals of Mathematics* by Emanuel Milman and Joe Neeman, with a comment on its relation to other recent research of the authors.
- ❑ We also provide summaries of several events:
  - Eva Miranda's Hardy Tour of the UK, from May 30 to September 21, 2023, in which she is scheduled to deliver nine lectures;
  - Ingrid Daubechies, winner of the Wolf Prize in Mathematics 2023;
  - The ICREA Academia distinction to Albert Atserias;
  - The taking office on March 28 of Jordi Guàrdia as new Dean of the **FME** (interviewed in NL04, pages 6-8).

Jaume Franch has recently finished his 8-year appointment as a dean of the Facultat de Matemàtiques i Estadística (**FME**) of the Universitat Politècnica de Catalunya (**UPC**). Before, he served as a vice-dean for almost 12 years under the deans Sebastià Xambó and Jordi Quer.

He obtained a "Llicenciatura en Matemàtiques" from the University of Barcelona in 1992. He got a position as full-time lecturer at **UPC** in 1993 and started his PhD in the Applied Mathematics program under the supervision of Enric Fossas, defending his thesis entitled *Flatness, tangent systems and flat outputs* in 1999. He became associate professor at **UPC** in 2002.

His research activity has always been mainly developed in the field of Control Theory, in particular in the study of the algebraic and geometric methods in nonlinear control and their applications to trajectory planning and control design. In collaboration with the research group led by Prof. Sunil K. Agrawal first at the University of Delaware and later on at Columbia University, where he has been a visiting scholar several times, he has applied his theoretical results to robotics and mechanical systems in general.

NL. ***At the end of your second 4-year period serving as dean of the FME, what institutional accomplishments would you like to highlight?***

We have known, for a long time, that our studies are at an excellence level, and during these years we have been able to establish that fact: all our studies have been accredited with the excellence mark. We must take into account that only about 10% of the Spanish university studies get this level of assessment. Besides that, I would like to highlight the following facts:

- The number of applications to our master degrees have increased significantly, especially in the Master in Mathematics. At present, our two masters are the ones with the highest number of applications and registered students in their fields in Spain.
- We have increased substantially the budget of the **FME** by working in two directions: on one hand, increasing the rates of our studies and being able to get recognition for that, and, on the other hand, by augmenting the income through collaborations with private companies. With this increased budget, we have been able to improve our facilities, an achievement which is very important under circumstances that demand sustained attention to technology advances.
- Thanks to Mireia Ribera, our administrative secretary, we have improved our communication to the community and the society at large. An example of this is "Notícies FME", a weekly newsletter where we make announcements and provide all kinds of information related to the **FME**.
- When I started the first term as a dean of the **FME** the number of people working as a technical staff at the **FME** had dropped dramatically. One of my goals was to get more personnel and we also succeeded in this.
- We are participating in other studies that are also important for us: the master for preparing high school mathematics' teachers, whose lectures are now taught at the **FME**, and the Data Science and Engineering degree, shared with the schools ETSETB and FIB.
- Finally, we supported the creation of **IMTech** from the very beginning: The **FME** provided a small financial help to assist in its first steps, and it is where the **IMTech** has its headquarters. This means, in particular, that the **FME** purveys administrative assistance and facilities for **IMTech** activities.

Before your mandates as a dean, you served as vice-dean for the three preceding mandates. What is your balance sheet about those experiences?

Actuallly, I served as vice-dean for the four preceding mandates, two with dean Sebastià Xambó, and two with dean Jordi Quer. I was very young when Sebastià Xambó asked me to be in his team, and I have great memories from that time. We did a lot of work. Jordi Quer also trusted me and, specially in the last three years, he empowered me a lot. So, for me it was a natural step to stand for the dean position after that.

How do you see the future of the FME? Do you have any advice for the new dean's team?

My best advice is to not spoil anything that is working well. Besides that, I think it is important to have some influence in the decisions made by **UPC** and to increase the collaboration with companies. It is also worth mentioning the change that will imply the increment in the number of students in the undergraduate studies in mathematics in our faculty. It was something necessary to do due to the increasing number of students willing to study mathematics and to the very high demand of our graduates from companies. Thankfully, this process has been done with the agreement of everybody at the **FME**, which will make easy to deal with the challenges that will come up with a larger number of students

We would also like to know about your teaching experience. What subjects did you teach? For how long? What are your favorite topics?

I started teaching Real Analysis at the **FME** 25 years ago. This is my favourite subject to teach, but I like to teach any other subject related to Calculus and Analysis. Besides that I also enjoy teaching Control Theory, my research area. Lately, I have become the responsible for the 4th year compulsory subject Mathematical Models in Technology. I am trying that this subject evolves towards a kind of subject where companies and research groups present some projects and the students work in them during all the term.

Let's talk about research. What drove you into problems in control theory and its applications? How is this commitment related to you PhD thesis?

When I joined **UPC** I was asked to td the PhD in one of the research lines of my department. I attended a course in Graph Theory, a course in Criptography and a course in Control Theory. The latter was the one I liked the most, and I asked Enric Fossas to supervise my PhD in this topic. My PhD was quite theoretical, and later on I have been able to do some applications of my results, mainly in the field of Robotics.

Administration service usually holds back research to a good extend. How has it been in your case?

During my years as a vice-dean I was able to maintain my research, mainly through yearly visits to the group headed by Sunil K. Agrawal, then at University of Delaware. But once I became dean I stopped doing research. When I do research I need to be focused in a problem, and I am not able to do research only in the very few free hours that you have when you are a dean.

Could you explain in more detail the nature and scope of your collaboration with Sunil K. Agrawal and his research group?

We met for the very first time 25 years ago, in my first international conference when I was still doing the PhD. He was interested in my theoretical results since they could be applied to design controllers for mechanical systems. He started inviting me every year and we published a good number of papers together in my ten visits to his group. Later on, we gave a graduate course together, first at University of Delaware and later at Columbia University.

Returning to a more active research life may need some adjusting period. What are your plans in that regard for the years ahead?

I am very excited about returning to a more active research life. I have started a collaboration with some researchers at IRI (Joint Research Center of the Spanish National Research Council (CSIC) and the Technical University of Catalonia (UPC)). We have applied for a research project to the Spanish Ministry. I would also like to collaborate with Prof. Agrawal again.

PhD highlights

Alberto Larrauri defended his PhD thesis *First Order Logic of Random Sparse Structures* on March 03, 2023.

The thesis was produced within the **UPC** doctoral program on Applied Mathematics and his advisor was Marc Noy.

Starting May 2023, he will be a postdoctoral researcher at the University of Oxford in the group of Standa Živný.

Thesis summary

In the case of graphs, first order (FO) logic speaks about adjacencies using quantification over vertices and Bolean connectives. A natural question in this area is: given a sequence of random graphs $\mathcal{G}_n$, does the limit probability that $\mathcal{G}_n$ satisfy P exist for every FO property P? When the answer is yes, we say that $\mathcal{G}_n$ satisfies a FO *convergence law*. As a follow-up to the last question, we can also ask what are the possible values of those limit probabilities. We may also be interested in studying the FO properties P that hold in $\mathcal{G}_n$ with high probability (w.h.p.). This thesis makes contributions in all these directions, focusing on *sparse* (as opposite to dense) random structures.

In the first part, published in [1], we look at a binomial model of random relational structures. We prove that a FO convergence law holds in this model when each relation is expected to grow linearly with the number of elements in the structure. Moreover, we show that the limit probability of each FO statement is given by an analytic expression in terms of the density of each relation. We extend this result to more complex models where symmetry and anti-reflexivity constraints are imposed on the random structure. Finally, we give an application our convergence results to the study of random SAT, suggested by Albert Atserias. A line of research here tries to establish the existence of efficient algorithms that accurately decide satisfiability of random CNF formulas with high probability. Our contribution here states, roughly, that if P is a FO statement that implies unsatisfiability of CNF formulas, then w.h.p. P does not hold in random CNF formulas with linear number of clauses.

In the second part we consider several random models where a FO convergence law is known to hold, and we study the set L of limiting probabilities of FO statements. More concretely, we consider the linear ranges of binomial random graphs and binomial random uniform hypergraphs [2], as well as of random graphs with given degree sequences. Our main result here is that L is dense in the whole interval [0,1] exactly when the random model in question contains some cycle with asymptotic probability exceeding $1/2$. Otherwise the closure of L is a finite union of closed intervals with some 'gaps' in between.

The final chapter is devoted to the study of probabilistic versions of FO preservation theorems. These are classical results that relate semantic and syntactic classes of FO sentences. For instance, Lyndon's Theorem states that any monotone sentence is equivalent to some positive sentence, and Łoś-Tarski Theorem states that any sentence closed under extensions is equivalent to some existential sentence. Crucially, these results are stated for the class of all structures (both finite and infinite), and fail when restricted to finite structures. We define probabilistic versions of those two preservation theorems where logical equivalence is replaced by almost sure equivalence, and ask whether those new results hold in several random graphs, obtaining multiple positive results. We consider the binomial random graph at the connectivity threshold, in the linear range, and in the sublinear range. Additionally, we also look at uniform random graphs from addable minor-closed classes.

Selected Publication: [2].

References

[1] Alberto Larrauri, Probabilities of first-order sentences on sparse random relational structures: An application to definability on random CNF formulas, *Journal of Logic and Computation* **31** (2021), 444-472.

[2] Alberto Larrauri, Tobias Müller, Marc Noy, Limiting probabilities of first order properties of random sparse graphs and hypergraphs, *Random Structures and Algorithms* **60** (2022), 506-526.

Outreach

The heat equation, by Luis A. Caffarelli
(Deparment of Mathematics, UT at Austin).

Facultat de Matemàtiques i Estadística
Universitat Politècnica de Catalunya
Lección inaugural del curso 2003-2004
22 de setiembre de 2003

La ecuación del calor

Professor Luis Caffarelli
Departament of Mathematics
University of Texas at Austin

On September 22, 2003, Luis A. Caffarelli delivered the inaugural lecture of the FME 2003-04 term. The image corresponds to the heading of the corresponding chapter in the booklet distributed by the FME on that occasion. Now, after twenty years, he has been awarded the 2023 Abel Prize and this NL rejoices in this great honor by offering a translation into English from the Spanish original of that lecture.

Fourier

The heat equation was proposed by Fourier in 1807—in his memoir on the propagation of heat in solid bodies.

In it, he also proposed the germ of what would become the theory of Fourier series.

So controversial was the latter that it took fifteen years, until 1822, for the Academy of Sciences to decide to publish it.

Mathematical models

The heat equation is a mathematical model (perhaps the simplest) that tries to describe the evolution of temperature in a solid body.

Let us consider, to simplify the presentation, an isolated metallic bar of length one ($0 \leqslant x \leqslant 1$), initially at zero temperature, which after a certain time, t_0, we have heated to a temperature $T(x, t_0)$ keeping its ends, $x = 0$ and $x = 1$, at zero temperature.

From that instant, t_0, we let the temperature $T(x, t_0)$ evolve freely and we are interested in a mathematical model that allows us to predict the temperature $T(x, t)$ for all x in the interval $[0, 1]$, for any future time (that is, for all $t > t_0$), from our knowledge of $T(x, t_0) = T_0(x)$ and from the fact that for $x = 0$ or $x = 1$ the temperature remains equal to zero.

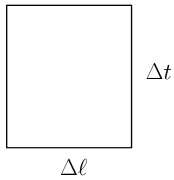
Naturally there is no "one model". There are infinitely many, depending on the precision and the range of values in which we intend it to be valid (high or low temperatures will change the behavior of the material, impurities could be relevant, etc.).

The model proposed by Fourier can be summarized as follows:

1. The (caloric) energy required to bring a piece of the bar of length $\Delta\ell$ from zero temperature to temperature T is proportional to $\Delta\ell \times T$ (i.e., the energy density, $e = kT$, is proportional to the temperature, with k a characteristic constant of the material).

2. Energy flows from areas of higher temperature to those of lower temperature. More precisely, the energy flux density is
$$f(x) = -\theta D_x T$$
(or $\boldsymbol{f}(x) = -\theta\nabla T$ in various dimensions), where θ is again a characteristic constant of the material.

3. The energy is conserved. If we take a piece of the bar, $\Delta\ell$, the energy contained in $\Delta\ell$ at the instant t_2 is equal to the energy that was in $\Delta\ell$ at the instant t_1 plus the "energy flux" that penetrated the extremes x_1, x_2 in the time interval from t_1 to t_2. Mathematically:
$$\int_{\Delta\ell} (e(x, t_2) - e(x, t_1))\, dx$$
$$= \int_{t_1}^{t_2} (-f(x_2, t) + f(x_1, t))\, dt.$$
If we draw the rectangle $\Delta\ell \times \Delta t$,



the first integral occurs at the top and bottom edges, while the second occurs at the sides. In order to compare them, we need to be able to write them in a common domain, namely the rectangle. We do this by taking derivatives:
$$\int_{\Delta\ell\times\Delta t} D_t e(x, t) dx dt = \int_{\Delta\ell\times\Delta t} -D_x f(x, t) dx dt.$$
Since this relationship must be verified for any rectangle, no matter how small, the integrands must necessarily be equal: $D_t e + D_x f = 0$.

Remembering the expressions for e and f as a function of T, we obtain the equation
$$k D_t T = \theta D_{xx} T.$$

In present day terms, the relationships 1) and 2) are called *constitutive laws*, and they establish specific relationships between the state variables, e, f, T and their derivatives, which depend on the characteristics of the state. materials etc. Relationship 3), on the other hand, is of a different nature, it is a conservation law, and establishes that certain quantities (mass, energy, etc.) are conserved through a process. That does not mean that they are point-wise constant. In a gas, for example, mass flows from one part to another. What a conservation law does is postulate the existence of a conserved variable, e, and a flow, $\boldsymbol{f}$, that satisfy
$$D_t e + D_x f = 0.$$
(or $D_t e + \operatorname{div}\, \boldsymbol{f} = 0$).

In short, writing a mathematical model consists of *choosing those state variables that are relevant to the phenomenon we want to describe*, finding (usually experimentally) *their constitutive laws*, and how they are conserved.

Existence and uniqueness

Perhaps the most important variation that these ideas have undergone today is in taking into account random effects.

Regardless of how good a mathematical model is at representing reality, it must have a minimum of internal consistency.

If the relationships we specify are excessive, they will generally be contradictory and our problem may not have a solution.

If they are too few, we may have many different solutions, when in reality we expect to have a single solution.

So Fourier's next step was to try to find a solution to the problem. Given the initial temperature, $T_0(x)$, and the condition

$$T(1,t) = T(0,t) = 0$$

for all $t > t_0$, it is a matter of proving that there is a unique function $T(x,t)$ that satisfies the equation $T_t = T_{xx}$ (we set $k = \theta = 1$).

Let's first try to find some solutions for particular T_0 of the form

$$T(x,t) = T_0(x)g(t).$$

This requires that

$$g'(t)T_0(x) = g(t)T_0''(x)$$

or that

$$\frac{g'(t)}{g(t)} = \frac{T_0''(x)}{T_0(x)} = \lambda \text{ constant}$$

(the only possible way for two functions of distinct variables to be equal is for them to be constants, since we can set t and vary x over all possible values).

Since $T_0(0) = T_0(1) = 0$, the only possible pairs are

$$T_0(x) = \sin(n\pi x)$$
$$g(t) = e^{-(n\pi)^2 t}.$$

But the problem we were considering is linear. Therefore, any combination of solutions

$$T(x,t) = \sum c_n \sin(n\pi x) e^{-(n\pi)^2 t}$$

is a new solution, with initial data

$$T_0(x) = \sum c_n \sin(n\pi x).$$

Fourier then proves that any function T_0 (say continuously differentiable, with $T_0(0) = T_0(1) = 0$) can be expressed that way, and gives a formula for the coefficients.

This is how Fourier analysis was born, so revolutionary that it took fifteen years for mathematicians of the time to accept that a series of highly oscillating functions such as $\sin(n\pi x)$ could represent, for example, an arc of a parabola or a polygonal.

Harmonic analysis

Fourier analysis has come to be called harmonic analysis. We can say that it consists of describing a function not by its special characteristics (where it is large, where it is small), but by the influence that each frequency ($\sin \lambda x$) has on its composition. As such, it occupies a fundamental place in everything related to wave theory, transmission of all kinds of signals, ultrasound image reconstruction, spectral analysis, etc.

In the last years a way of decomposing functions into "elementary chunks" that describe the oscillatory properties of the function simultaneously in physical space (the variable x) and frequency space (the variable n or λ) has acquired great prominence.

These elementary chunks are called *wavelets* and have revolutionized image compression, data transmission, etc.

The Gaussian kernel and random walks

There is actually a more convincing way of representing the solution $T(x,t)$, one that more clearly exhibits the qualitative properties of heat propagation. It consists of initially putting "point masses".

Suppose now that the bar is infinite, it is at zero temperature and we manage to place a "point mass" of one heat unit at the origin and at the instant t_0. In other words, we were able to concentrate a quantity $c = 1$ of heat energy at the origin so quickly that it is instantaneous for our time scale.

How does the temperature evolve next?

A little self-similarity analysis: if $T(x,t)$ is a solution of the equation, so is $aT(bx, b^2t)$, which allows us to calculate that in this case

$$T(x,t) = \frac{1}{(\pi t)^{1/2}} e^{-x^2/4t} = G(x,t).$$

This is the Gaussian kernel ("the bell"), or error dispersion formula.

If we translate the point mass to x_0, the new formula is

$$T(x,t) = G(x - x_0, t),$$

since the equation is translation invariant .

If we superimpose point masses of intensities c_i on the points x_i,

$$T(x,t) = \sum c_i G(x - x_i, t)$$

and finally, for an energy density $e = T_0(x)$,

$$T(x,t) = \int G(x_0, t)\, T(x_0)\, dx_0.$$

This representation immediately tells us, among other things, that:

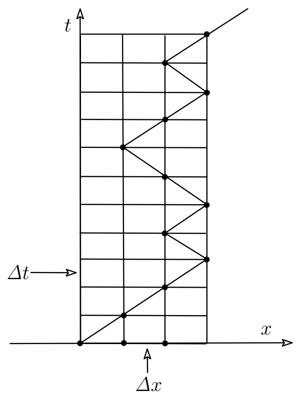
a) If the original temperature is positive, it remains positive.

b) The effect of any change in temperature is felt instantly throughout the bar.

c) The temperature T_0 can be highly discontinuous and an instant later it becomes regular.

But what is the relationship between the heat equation and error propagation?

Suppose that at the instant t_0 we are standing at the origin. We flip a coin; if heads, we take one step, Δx, to the right. If tails, to the left. Every interval Δt, we repeat the operation.



What is the probability $u(x,t)$ that at time t we find ourselves in position x?

It seems complex to calculate, but we can see that at the instant $t - \Delta t$ we were either at $x + \Delta x$ or at $x - \Delta x$ and that from there we moved with probability ½ to (x,t), that is,

$$u(x,t) = \frac{1}{2}\left(u(x + \Delta x, t - \Delta t) + u(x - \Delta x, t - \Delta t)\right)$$

or

$$\begin{aligned} u(x,t) - u(x, t - \Delta t) = \tfrac{1}{2}\big(&u(x + \Delta x, t - \Delta t) \\ &+ u(x - \Delta x, t - \Delta t) \\ &- 2u(x, t - \Delta t)\big). \end{aligned}$$

Everything now depends on the balance between Δt and $(\Delta x)^2$. If we choose $\frac{(\Delta x)^2}{\Delta t} = 1$, we can divide both sides by Δt and we get:

$$\frac{\Delta u}{\Delta t} = \frac{\Delta^2 u}{(\Delta x)^2},$$

which is a discrete form of the heat equation.

That is, in the limit $\Delta t \to 0$, u converges to the solution of the heat equation. But at the initial instant, we are standing at the origin with probability 1, that is,

$$u(x,t) = G(x,t).$$

This is a version of the central limit theorem which says that if we independently repeat n times the same zero expectation experiment X_i, then the probability distribution of

$$X = \frac{\sum X_i}{\sqrt{n}}$$

converges to a Gaussian.

Nonlinear diffusions

That is why a heat-type equation is often called a diffusion equation. Diffusion equations appear in various fields. For example, in population dynamics the energy density e is replaced by the population density σ, and one of the many reasons why a population migrates is to go to areas of lower density, that is, the population flow has the form

$$f = -\nabla\sigma + \cdots \text{ (other reasons)}$$

and therefore the corresponding equations will be of the form

$$D_t\sigma = \Delta\sigma + \cdots$$

Or, in an epidemic, the probability of infection at a place x, at an instant of time t, depends monotonically on the probabilities of adjacent points a few hours earlier. This gives rise, infinitesimally, to an equation of the form

$$D_t e = F(D_x^2 e, \nabla e)$$

where e is the expectation of infection at x, t.

In a viscous fluid, particles adjacent to a given one try to “drag” or “slow down” it if it is slower or quicker, respectively, then the others.

The point I want to emphasize is that, in all these phenomena, the “diffusion” or “viscosity” term induces a process of “flattening” or “averaging” of the state variables that characterizes diffusive or viscous problems.

The influence of the theory of “parabolic equations” is today immense, in fluid equations (Navier–Stokes, flow in porous media, phase change equations), in optimal control theory and game theory (totally non-linear equations), modeling of population dynamics, epidemiology, mathematics of finance, etc.

Chronicles

A personal glimpse on Professor Luis Caffarelli on the occasion of the Abel Prize 2023

by Juan Luis Vázquez (RAC, UAM, UCM)

Received on March 30, 2023.

(This text is an English version, slightly edited, of the article just published in Spanish as [1]).

A week ago, the Norwegian Academy of Science and Letters awarded the Abel Prize 2023 to the Argentine-American mathematician Luis Ángel Caffarelli in an announcement made in Oslo. Luis is well known in the international mathematical community for his fundamental contributions to the theory of regularity for nonlinear PDEs (Partial Differential Equations), which include free boundary problems, the equations of viscous fluids, the Monge-Ampère equation, optimal transport equations and many other topics. After last week's brief scientific presentation [2] today we would like to point out some more personal aspects of the first Spanish-speaking Abel prize winner and also highlight his intense relationship of many years with Spain.

Luis Caffarelli was born in 1948 in Buenos Aires, Argentina, and has lived in the US since 1973. His early American years were spent in Minnesota, a distant region of the Midwest that brings back such fond memories to so many Spanish scientists of my generation. Luis rose to fame at the end of that decade for his surprising work on the regularity of the so-called free boundaries, known to the public today largely thanks to his work. In fact, he surprised the whole world with the article [3], a work whose novelty and brilliance was the basis of his future fame. Indeed, his initial studies on the Obstacle Problem, [4], are already a classic in pure and applied mathematics. Another name to remember in this topic is the Stefan Problem, which models, among other applications, the evolution of the ice-water system with its separation interface.

A second hit came in 1982 with the article [5], produced during his 2-year stay at the mythical Courant Institute, where he collaborated with Robert Kohn and Louis Nirenberg. The latter, also an Abel Prize winner (2015), was his supporter in those years and was his friend for life. Years go by and this beautiful result, the CKN theorem, continues to stand out as the last great contribution made in the study of the regularity of the solutions of the Navier-Stokes equations for viscous fluids, which is one of the problems of the Millennium of the Clay Foundation.

The eighties were a prodigious time for Luis and a cascade of articles with various collaborators marked the breadth and depth of his mathematical ability and established him as the best worldwide representative of the legacy of the great Ennio de Giorgi on how to study the regularity inherent in the solutions of the problems of the so-called Calculus of Variarions, which is today a main branch of mathematics. Luis added to the program of the great Ennio the free boundary problems that had bewildered the best experts during the 60s and 70s due to their intricate combination of analytical and geometric difficulties.

Over the years, Luis has been and is one of the world's leading experts in the field of nonlinear Partial Differential Equations. The PDEs are a discipline established within the body of Mathematics in the 18th-19th centuries, one of the many brilliant daughters of Calculus, and today it is experiencing one of its golden moments due to the enormous influence of its results and techniques in the most diverse directions, ranging from the fundamental equations of Physics to the mathematical models of various engineering and other sciences of growing social interest.

The mathematics of the 20th century have been excellent in the study of non-linear processes, which represent a stage of difficulty higher than the study of linear processes. Although Nature has had the good taste of resorting to linear processes for many of its basic models (such as the propagation of waves or heat, and also the basic equations of quantum theory), today it is well known that many of the fundamental processes of Science are non-linear, and understanding that added difficulty is the glory and the cross of the current mathematical profession. Briefly stated, nonlinearity is an infinite source of complexity.

From 1986 to 1996 Luis was a permanent member of the Institute for Advanced Study of Princeton. He was later a professor at the Courant Institute of Mathematical Sciences in New York, before joining the University of Texas at Austin in 1997 as the Sid Richardson Chair in Mathematics. Early recognized as a scientific referential in the US, he has been a member of the US National Academy of Sciences since 1991. But he was no less recognized in Spain where he was bestowed a Doctorate Honoris Causa by the Autonomous University of Madrid in 1992.

A series of relevant prizes show the impact of his scientific contributions. In this century, for instance, we count the Rolf Schock Prize, from the Royal Swedish Academy of Sciences (2005); the Leroy P. Steele Prize "for Lifetime Achievement" from the American Mathematical Society (AMS, 2009); the very prestigious Wolf Prize in Mathematics (2012); the Solomon Lefschetz Medal, from the Mathematical Congress of the Americas (2013); a Leroy P. Steele Prize again, this time "for Seminal Contribution to Research", AMS (2014, shared with Robert Kohn and Louis Nirenberg); as well as the Shaw Prize in Mathematics, (2018). Since 2015 he has been a foreign member of the Spanish Royal Academy of Sciences, where I had the honor of introducing him.

Luis Caffarelli has had an enormous influence on the development of partial differential equations in several countries, mainly the US, Argentina, Spain, Italy and Greece (to make the list short). His collaboration with Spanish authors dates from the 1980-90s. After meeting him at a conference on free boundaries in Italy in the summer of 1981, I enjoyed long stays in Minnesota, where Luis Caffarelli was a professor at the time, and contact with him was strengthened by his regular visits to Spain. The collaborations carried out include a good number of Spanish co-authors, in particular Xavier Cabré (with whom he wrote a famous book, [6]; see Reviews in this issue), Antonio Córdoba, Ireneo Peral (1946-2021), Rafael de la Llave, Fernando Soria, María del Mar González and the author of this glimpse, among others. These collaborations with Spanish authors deal with regularity problems of highly nonlinear equations and phase change problems and free boundaries, issues in which his leadership is uncontested. I am the co-author of 8 articles with Luis and my best-known book [7] is dedicated to him because of the many pages that refer to his ideas.

Luis Caffarelli and Juan Luis Vázquez at work (UAM, 2017)

Luis has participated in numerous courses and schools in Spain, particularly in the Summer Courses of the UIMP in the incomparable framework of the Palacio de la Magdalena de Santander, courses that Luis inspired and that UAM co-sponsored, a university that he visited with some frequency from 1986 to 2017. I am witness to this since I participated in all these courses, which had the strong support of the rector Ernest Lluch (1937-2000), eminent patron of science at the UIMP. The strong international orientation responded to Luis's ideas but was not easy to implement with the existing local rules. The series of courses of the 80s and 90s continued under the rector Salvador Ordóñez between 2010 and 2015. And we must not forget Luis' strong ties with Barcelona and Granada, for example. In recent years Luis has been involved with us in the study of anomalous diffusion problems with non-local operators, another source of friendships, travels, and mathematical anguish and pleasure. His most notable paper on this subject is [8], a true best seller in the area.

Luis has written more than 320 mathematical articles with more than 130 collaborators, from the most reputed authors to bright young people looking for a future in mathematics. He has advised more than 30 PhD students. The depth of his ideas, the breadth of his subject matter, coupled with his generosity and easy manner, have cemented his fame on all continents, and by all continents we mean people we know on all of them. *Luis alone has been a non-linear university for the world*.

In addition, he has the well-deserved reputation of being a great cook in the Argentine-Italian tradition. Congratulations, Maestro.

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Reviews

Books

Fully Nonlinear Elliptic Equations

by Luis Caffarelli and Xavier Cabré, [18].

Reproducing the review by John Urbas in the Book Reviews of the Bulletin of the AMS, with an annex by the NL editors collecting a few follow up notes.

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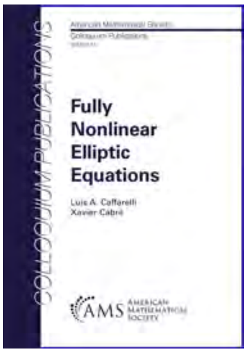

One of the major advances in the theory of partial differential equations during the last twenty years has been the development of techniques for studying fully nonlinear second-order elliptic equations. These are equations of the general form

$$F[u] = F(D^2u, Du, u, x) = 0 \quad (1)$$

where F is nonlinear with respect to D^2u as well as possibly with respect to u and Du. Ellipticity means that F is a monotone function of the second derivative variables, in the sense that for any $(M, p, z, x) \in \mathbf{S}^{n\times n} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$, where $\mathbf{S}^{n\times n}$ denotes the space of $n \times n$ real symmetric matrices, we have

$$F(M + N, p, z, x) > F(M, p, z, x) \quad (2)$$

for any positive definite $N \in \mathbf{S}^{n\times n}$. In addition, F is said to be uniformly elliptic if for any positive $N \in \mathbf{S}^{n\times n}$ we have

$$\lambda\|N\| \leqslant F(M + N, p, z, x) - F(M, p, z, x) \leqslant \Lambda\|N\| \quad (3)$$

for all $(M, p, z, x) \in \mathbf{S}^{n\times n} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$, where λ and Λ are some positive constants, called the ellipticity constants of F. Examples of such equations are the Bellman equation

$$F[u] = \inf_{\alpha\in\mathcal{A}} \{L_\alpha u - f_\alpha(x)\} = 0 \quad (4)$$

and Isaacs' equation

$$F[u] = \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \{L_{\alpha\beta} u - f_{\alpha\beta}(x)\} = 0 \quad (5)$$

where each L_α, $L_{\alpha\beta}$ is a linear elliptic operator of the form

$$L = a_{ij}(x)D_{ij} + b_j(x)D_i + c(x). \quad (6)$$

These are uniformly elliptic if each L_α, $L_{\alpha\beta}$ is uniformly elliptic with ellipticity constants independent of α and β. Bellman and Isaacs' equations arose originally in stochastic control theory and stochastic games theory, but it turns out that many equations arising elsewhere can also be written in these forms.

Two main strategies have been developed for solving fully nonlinear elliptic equations. One approach is to prove the existence of classical solutions of, say, the Dirichlet problem in a smooth bounded domain $\Omega \subset \mathbf{R}^n$ directly using the continuity method. For this, one needs to prove a priori estimates for solutions in the space $C^{2,\alpha}(\overline{\Omega})$ for some $1 < \alpha < 1$; i.e., one needs to bound u and its derivatives up to second order as well as the α Hölder seminorm of the second derivatives. This approach has led to the existence of classical solutions for a wide variety of fully nonlinear elliptic equations subject to various boundary conditions, but without doubt the central results are the interior second derivative Hölder estimates of Evans [6, 7] and Krylov [12] and the corresponding global estimate of Krylov [13]. These results in turn depend on the Harnack inequality for linear elliptic equations in nondivergence form due to Krylov and Safonov [15, 16].

The second approach is to prove the existence of some kind of generalized solutions and then to establish their uniqueness and regularity. The concept of generalized solution which has evolved in the work of Evans [4, 5], Crandall and Lions [3], and Lions [17] is that of viscosity solution, although for specific classes of equations there are also other notions. A continuous function u defined on a domain in $\mathbf{R}^n$ is said to be a *viscosity subsolution* (respectively, *viscosity supersolution*) of the elliptic equation

$$F(D^2u, x) = 0 \quad (7)$$

if for any C^2 function ϕ on Ω and any local maximum (respectively, minimum) $x_0 \in \Omega$ of $u - \phi$ we have $F(D^2\phi(x_0), x_0) \geqslant 0$ (respectively, $\leqslant 0$). A *viscosity solution* is a continuous function that is both a viscosity subsolution and supersolution. The point is that the test function ϕ should satisfy the inequalities which would hold by virtue of the maximum principle and the ellipticity of the equation if u were a C^2 solution. The notion of viscosity solution turns out to be very useful, because it is stable under locally uniform convergence of both u and F and because existence and uniqueness results for such solutions can be proved under very general conditions, and in particular for equations for which the existence of classical solutions is not known (and perhaps not true), such as Isaacs' equation in dimensions greater than two.

A major breakthrough in the theory of viscosity solutions was made by Jensen [11], who proved a comparison principle which implied the uniqueness of viscosity solutions of the Dirichlet problem for (7), at least for F independent of x. Later refinements allowed this assumption to be relaxed. Using these results, Ishii [9, 10] observed that the existence of viscosity solutions of the Dirichlet problem followed from the Perron method.

Finally we come to the main topic of this book, which is the regularity theory for viscosity solutions of uniformly elliptic equations of the general form

$$F(D^2u, x) = f(x). \quad (8)$$

It is based on a series of lectures given at New York University in 1993. The central results are the $W^{2,p}$, $C^{2,\alpha}$, and $C^{1,\alpha}$ estimates of the first author [1, 2]. To describe these, it is probably best to follow the book and to recall the corresponding estimates for linear equations. Let u be a bounded solution of the uniformly elliptic equation

$$Lu = a_{ij}(x)D_{ij}u = f(x) \quad (9)$$

in the unit ball $B_1 \subset \mathbf{R}^n$. Then the following are true:

(i) (Cordes-Nirenberg type estimates) Let $0 < \alpha < 1$ and suppose that

$$\|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \leqslant \delta(\alpha)$$

for sufficiently small $\delta(\alpha) > 0$. Then $u \in C^{1,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leqslant C\big(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\big)$$

(ii) (Schauder estimates) If a_{ij} and f belong to $C^\alpha(\overline{B}_1)$, then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leqslant C\big(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_1)}\big)$$

(iii) (Calderón-Zygmund estimates) If a_{ij} are continuous in B_1 and $f \in L^p(B_1)$ for some $1 < p < \infty$, then $u \in W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leqslant C\big(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\big).$$

These estimates are obtained by first deriving the estimate for solutions of $\Delta u = f$ and then using a perturbation technique. The idea in the nonlinear case is similar: if one has suitable existence results and interior estimates for solutions of the "constant coefficient" equation

$$F(D^2u, x_0) = f(x_0), \tag{10}$$

then under certain assumptions on F and f one can derive $C^{1,\alpha}$, $C^{2,\alpha}$, or $W^{2,p}$ estimates for viscosity solutions of (8) by a perturbation argument. The basic assumption on F required to carry out this procedure is that the quantity

$$\beta(x, x_0) = \sup_{M \in \mathbf{S}^{n\times n}} \frac{|F(M, x) - F(M, x_0|}{\|M\|}$$

is sufficiently small if $|x - x_0|$ is small. The precise nature of this smallness condition and the estimates required for solutions of (9) depend on which estimate one is considering, but in each case the smallness of $\beta(x, x_0)$ is measured in the L^n norm rather than the L^∞ norm. The techniques thus give improved versions of the classical linear estimates stated above.

The book begins with some preliminary material concerning tangent paraboloids and second-order differentiability. In Chapter 2 viscosity solutions are introduced, and the class $\mathcal{S}(\Lambda, \lambda, f)$ of "all viscosity solutions of all elliptic equations of the form (8) with ellipticity constants Λ and λ" is defined using Pucci's extremal operators. This is important in what follows because it allows the authors to avoid the traditional Bernstein method of differentiating the equation to obtain linear differential inequalities for the derivatives of the solution. Clearly, this procedure is not possible if F and u are not sufficiently smooth.

Chapters 3 and 4 deal with two crucial tools from the linear theory. These are the Alexandroff-Bakelman-Pucci estimate and maximum principle and the Harnack inequality of Krylov and Safonov. These are proved for viscosity solutions rather than classical solutions, so the proofs are a little more complicated in certain parts than those presented elsewhere, for example in [8] or [14]. An important consequence of the Harnack inequality is the C^α interior regularity of solutions of (8).

The following two chapters deal with the existence and uniqueness results and estimates for solutions of uniformly elliptic equations of the form

$$F(D^2u) = 0 \tag{11}$$

which are necessary to carry out the perturbation procedure mentioned above. A proof of Jensen's comparison principle for viscosity solutions of (11) is given, but the existence of solutions of the Dirichlet problem is not proved. It is only remarked that the Perron method can be used for this once one has a comparison principle. A $C^{1,\alpha}$ interior estimate for solutions of (11) is proved, and in Chapter 6 a version of the Evans-Krylov $C^{2,\alpha}$ interior estimate for C^2 (in fact, even $C^{1,1}$) solutions of concave equations of the form (11) is presented. In addition, a new proof of the $C^{1,1}$ interior regularity of viscosity solutions of such equations is given.

In the following two chapters these estimates are combined with delicate perturbation arguments to obtain the $W^{2,p}$, $C^{2,\alpha}$, and $C^{1,\alpha}$ estimates mentioned above. This is the most technical part of the book and cannot be described in detail here. The key idea, however, is to consider paraboloids of the form $P(x) = u(x_0) + l(x - x_0) \pm \frac{1}{2}M|x|^2$, where l is a linear function, and to show that the measure of the complement in $B_{1/2}$ of the set of points x_0 at which there is such a paraboloid touching the graph of u from above (or from below) must decay sufficiently quickly as M gets large. This gives control of the distribution function of second-order difference quotients of u, leading to $W^{2,p}$ estimates. The $C^{2,\alpha}$ (respectively, $C^{1,\alpha}$) estimates are proved by showing that the existence and regularity results for solutions of (10) imply that the solutions of (8) are well approximated by quadratic polynomials (respectively, affine functions).

In the final chapter the authors present an alternative proof of the $C^{1,1}$ interior estimate for smooth solutions of concave equations of the form (11). In addition, they describe the proof of the classical solvability of the Dirichlet problem for such equations using the continuity method. Most of the necessary estimates are proved in detail, but the proof of Krylov's boundary gradient Hölder estimate is omitted.

The book is well written, with the arguments clearly presented. There are helpful remarks throughout the book, and at several points the authors give the main ideas of the more technical proofs before proceeding to the details. No previous knowledge of viscosity solutions is assumed, but readers who are not familiar with the existence and uniqueness theory of viscosity solutions will probably want to consult other sources for this, as these aspects are not covered in detail. The book will certainly be of interest to researchers and graduate students in the field of nonlinear elliptic equations.

2023 Annex by the NL editors

The review [19] summarizes the contents of the book and ends with this appraisal: "This book provides a self-contained and detailed presentation of the regularity theory for viscosity solutions of fully nonlinear elliptic equations as developed in the last decade. It can be highly recommended to researchers as well as to graduate students who are interested in this area."

In [20], which also provides a detailed review, it is asserted that: "The book [...] is clearly written and gives an excellent impression of progress with the regularity theory for equations of types (1) $[F(D^2u, x) = f(x)]$ and (2) $[F(D^2u) = 0]$; it highlights in a most useful way some significant recent advances (in which the authors have been very active) which play a dominant part in the theory. [...] All in all, the book marks an important stage in the theory of nonlinear elliptic problems. Its timely appearance will surely stimulate fresh attacks on the many difficult and interesting questions which remain."

The book has been a bestseller within the AMS bookstore at different periods. As of this writing (March 31st, 2023), it has 955 citations in MathSciNet and 1.726 in Google Scholar.

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Reviews

Articles

The Gaussian Double-Bubble and Multi-Bubble Conjectures

by Emanuel Milman and Joe Neeman, [1].
Reviewed by Joaquín Pérez.

For the Euclidean space $\mathbb{R}^n$, the multiple-bubble problem consists of finding the smallest area that encloses and separates $q \in \mathbb{N}$ regions of prefixed positive volumes. Therefore, the classical isoperimetric problem is obtained for $q = 1$ and the double-bubble problem is the case $q = 2$. The double-bubble problem was solved in $\mathbb{R}^3$ by M. Hutchings, F. Morgan, M. Ritoré and A. Ros, [2], and for general dimension n by B. Reichardt, [3]. The solution to this problem is the standard spherical double-bubble given by three spherical pieces that intersect each other forming a dihedral angle of $2\pi/3$ along their intersection, a configuration that is easily produced when blowing soap bubbles. For the q-bubble case, $3 \leqslant q \leqslant n+1$, the optimal way to enclose and separate q regions of prescribed volume in $\mathbb{R}^n$ is conjectured to be the standard spherical q-bubble, i.e., a general version of the standard double bubble above. This question is still open.
The case of the Gaussian space is a relevant one related to several questions in PDEs, Probability, Banach spaces and Geometry. The Gaussian space $(\mathbb{R}^n, \gamma^n)$ is the Euclidean space endowed with the probability measure

$$\gamma^n = (2\pi)^{-n/2} \exp\left(-|x|^2/2\right)\, dx.$$

For $q = 2$, the isoperimetric problem in the Gaussian space and its solution was obtained in the 1970s by V.N. Sudakov and B.S. Tsirelson, [4], and independently by C. Borell, [5]. They proved that a hyperplane dividing the space in the two prescribed Gaussian measures is a minimizer for this problem. Later, E.A. Carlen and C. Kerce, [6], proved that halfspaces are in fact the unique minimizing clusters minimizers for the Gaussian isoperimetric inequality.
As in the Euclidean case, for $3 \leqslant q \leqslant n+1$ there is a natural Gaussian multi-bubble conjecture, which states that the least perimeter way to decompose the Gaussian space $(\mathbb{R}^n, \gamma^n)$ into q cells is the symmetric simplicial cluster given by the Voronoi cells determined by q equidistant points. In this paper the authors resolve the conjecture:

Gaussian Multi-Bubble Theorem: *For all $2 \leqslant q \leqslant n+1$, (symmetric) simplicial q-clusters are the unique minimizers of the total Gaussian perimeter in $(\mathbb{R}^n, \gamma^n)$ among all q-clusters of prescribed Gaussian measures.*

To understand the above statement, we need some concepts:

1. A *q-cluster* $\Omega = (\Omega_1, \ldots, \Omega_q)$ is a q-tuple of pairwise disjoint Borel subsets $\Omega_i \subset \mathbb{R}^n$ (called *cells*), of finite Gaussian perimeter and such that $\gamma(\mathbb{R}^n \setminus \cup_{i=1}^q \Omega_i) = 0$ (cells are not necessarily connected).

2. The total Gaussian perimeter of a q-cluster Ω is

$$P_\gamma(\Omega) := \frac{1}{2}\sum_{i=1}^{q} P_\gamma(\Omega_i),$$

where the Gaussian perimeter of a Borel set $U \subset \mathbb{R}^n$ is

$$P_\gamma(U) = \sup\left\{ \int_U (\mathrm{div} X - \langle \nabla W, X\rangle) d\gamma \mid X \in C_c^\infty(\mathbb{R}^n, T\mathbb{R}^n),\ \|X\| \leqslant 1 \right\}.$$

3. The Gaussian measure of a q-cluster Ω is the element $\gamma(\Omega)$ in the $(q-1)$-dimensional simplex $\Delta^{(q-1)} = \{v \in \mathbb{R}^q \mid v_i \geq 0, \sum_{i=1}^q v_i = 1\}$ given by

$$\gamma(\Omega) = (\gamma(\Omega_1), \ldots, \gamma(\Omega_q)) \in \Delta^{(q-1)}.$$

4. A *simplicial q-cluster* is the set of Voronoi cells of q equidistant points $x_1, \ldots, x_q \in \mathbb{R}^n$. These Voronoi cells are

$$\Omega_i = \mathrm{int}\left\{ x \in \mathbb{R}^n \mid \min_{j=1,\ldots,q} \|x - x_j\| = \|x - x_i\| \right\}, \quad i = 1, \ldots, q.$$

Therefore, the cells of a simplicial 2-cluster are precisely half-spaces in $\mathbb{R}^n$, and and the single-bubble Gaussian conjecture for $q = 2$ holds by the classical Gaussian isoperimetric inequality. For $q = 3$ we have the double-bubble Gaussian case (with prescribed Gaussian volume pair $v \in \mathrm{int}\, \Delta^{(2)}$), whose minimizer is given by three half-hyperplanes meeting along an $(n-2)$-dimensional subspace with dihedral angles of $2\pi/3$.
The proof of this groundbreaking result is based on the application of the maximum principle for a certain fully non-linear second-order elliptic PDE that involves the *Gaussian isoperimetric profile* $I^{(q-1)} : \Delta^{(q-1)} \to \mathbb{R}$ given by

$$I^{(q-1)}(v) = \inf\left\{ P_\gamma(\Omega) \mid \Omega \text{ is a } q\text{-cluster with } \gamma(\Omega) = v \right\}.$$

Remarks. The authors of the paper under review posted [7] in May 2022. This work deals with bubbles in $\mathbb{R}^n$ and $\mathbb{S}^n$ (Euclidean case), is not cited in [1], and seemingly has not been published yet. It is reviewed in AMR by F. Morgan.

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Events

Eva Miranda's Hardy tours

As announced on page 6 of the Issue 505 of the LMS Newsletter (March 2023), **IMTech** researcher Eva Miranda will deliver the Hardy lectures 2023: "The London Mathematical Society is pleased to announce that the LMS Hardy Lecturer 2023 is Professor Eva Miranda (**UPC** and CRM–Barcelona).

The Hardy Lectureship was founded in 1967 in memory of G. H. Hardy in recognition of outstanding contribution to both mathematics and to the Society. The Hardy Lectureship is *a lecture tour of the UK by a mathematician with a high reputation in research*. Professor Miranda will undertake three lecture tours of the UK between May and September 2023, which will include the Hardy Lecture at the Society Meeting on Friday 30 June in London. Further details about the Hardy Lecture Tour 2023 are available on the website Hardy Lectureship."

G.H. Hardy, LMS President 1926-1928 and 1939-1941

E. Miranda (UPC Barcelona) Hardy Lecturer 2023

For details about dates, institutions, lecture titles and abstracts of the nine scheduled lectures, we refer to Miranda's Hardy Tour Web page. Here is a short summary:

First tour

- ❑ May 30, Cambridge, Department of Mathematics: *Counting periodic orbits*.
- ❑ June 1, London, Royal Institution: *From Alan Turing to contact geometry: towards a "Fluid computer"*.

  This activity will be complemented by the art installation *It from Bit: Psychedelic Fluids*, by Binghui Song and Tianxiang Shi.

  For more details visit the lecture announcement.

Second tour

- ❑ June 26, Birmingham, Department of Mathematics: *Desingularizing singular symplectic structures*.
- ❑ June 28, Warwick, Department of Mathematics: *Euler flows as universal models for dynamical systems*.
- ❑ 30 June, London, Mary Ward house, Hardy Lecture: *From Alan Turing to Fluid computers: Explored and unexplored paths*. This lecture is scheduled as part of the General meeting of the LMS and will be preceded by a warm-up lecture by Sir Roger Penrose.
- ❑ July 4, Oxford University: *Singular Hamiltonian and Reeb Dynamics: First steps*.
- ❑ July 6, Loughborough University: *Action-angle coordinates and toric actions on singular symplectic manifolds*

Third tour

- ❑ September 19, Edinburgh University: *From Symplectic to Poisson manifolds and back*.
- ❑ September 21, University of Glasgow: *Quantizing via Polytope counting: Old and new*

Ingrid Daubechies wins the 2023 Wolf Prize in Mathematics

Excerpts from [1]:

■ The 2023 Wolf Prize in Mathematics is awarded to Ingrid Daubechies, Duke University, USA, *for work in wavelet theory and applied harmonic analysis*.

■ Her research has revolutionized the way images and signals are processed numerically, providing standard and flexible algorithms for data compression. This has led to a wide range of innovations in various technologies, including medical imaging, wireless communication, and even digital cinema.

■ The Wavelet theory, as presented by the work of Professor Daubechies, has become a crucial tool in many areas of signal and image processing. For example, it has been used to enhance and reconstruct images from the early days of the Hubble Telescope, and to detect forged documents and fingerprints. In addition, wavelets are a vital component of wireless communication and are used to compress sound sequences into MP3 files.

■ Beyond her scientific contributions, Professor Daubechies also advocates for equal opportunities in science and math education, particularly in developing countries. As President of the International Mathematical Union, she worked to promote this cause. She is aware of the barriers women face in these fields and works to mentor young women scientists and increase representation and opportunities for them.

For more information on her life, work, and academic accomplishments, see the web pages [2–5]. The citation of [5] (2013):

■ This year's BBVA Foundation Frontiers of Knowledge Award in Basic Science goes to two mathematicians: Professor Ingrid Daubechies for her work on wavelets, and Professor David Mumford for his contributions to algebraic geometry and to the mathematics of computer vision. These works in pure mathematics have strongly influenced several fields of application, ranging from data compression to pattern recognition.

■ Professor Daubechies is a leader in theoretical signal processing, with pioneering contributions to the theory and application of wavelets and filter banks. Her work resulted in a new approach to data compression, with a strong impact on a multitude of technologies, including efficient audio and video transmission and medical imaging.

■ Professor Mumford introduced the modern approach to algebraic geometry into a classical area through his work on geometric invariant theory. He also applied tools of variational calculus to the theory of vision and developed statistical models for imaging and pattern recognition. His work has had a lasting impact in both pure and applied mathematics.

References

[1] Wolf Foundation, **Ingrid Daubechies**: Wolf Prize Laureate in Mathematics, 2023. 2023 Wolf Prize in Mathematics.

[2] Scholars@Duke, 2023. Ingrid Daubechies.

[3] Duke University, Duke Professor Wins One of the Most Prestigious Awards in Mathematics, 2023. Duke today.

[4] Wikipedia, Ingrid Daubechies, 2023. Ingrid Daubechies.

[5] FBBVA, Ingrid Daubechies, Frontiers of Konwledge Laureate. Frontiers of Knowledge Awards.

IMTech member Albert Atserias awarded ICREA Academia distinction

The ICREA Academia program was launched in 2008 with the aim of promoting and rewarding the research excellence of professors at public universities in Catalunya, who are in an active and expansive phase of their research career. Recognized researchers receive a research grant of 40,000 euros a year for a period of five years to promote their research. In the 2022 call, a total of six UPC researchers have been recognized with the ICREA Academia Prize.

Albert Atserias is full professor at the UPC Department of Computer Science, and a member of IMTech and Centre de Recerca Matemàtica. He was the principal investigator of the ERC-CoG project (2015–2020) AUTAR: A Unified Theory of Algorithmic Relaxations, funded by the research funding agency of the European Commission (ERC). His groundbreaking result with Moritz Müller on the computational complexity of proof search, published in the J. of the ACM in 2020, was outlined in a Research focus published in NL01 (pages 6-7).

His research focuses on logic and the theory of computation. His expertise is on computational and algorithmic complexity, more particularly, on descriptive complexity and proof complexity. During the next five years his efforts will pursue the research line on algorithmic complexity that aims at establishing variants of the most important conjectures in the area, such as the famous $P \neq NP$ or $P = BPP$, in non standard models of computation. The goal of the research is to establish these conjectures in non standard models which are however realistic enough so that it is still possible to deduce some of its most relevant consequences for the foundations of algorithmic complexity, for instance the existence of generators of guaranteed pseudorandom bits. Computational complexity studies the limitations of algorithms: what they can do and, especially, what computers can't do. The conjecture predicts that for solving some algorithmic tasks we will need more resources than we can ever have. The conjecture $P \neq NP$ is one of the seven millennium problems established by the Clay Foundation.

Events

Jordi Guàrdia, new Dean of the FME
Photos: Conchi Martínez

Professor Jordi Guàrdia Rúbies took office on March 28 as Dean of the Faculty of Mathematics and Statistics (FME) of the **UPC**. He is associate professor of the **UPC** Departament of Mathematics and succeeds Jaume Franch, who has been Dean of the **FME** for the last eight years (see the interview with him in this issue, and the inteview with Jordi Guàrdia in NL04, pages 6-8).

The ceremony was held at the **FME** and was chaired by the rector of the UPC, Daniel Crespo, accompanied by Jaume Franch, Jordi Guàrdia, and the general secretary of the UPC, Ana B. Cortinas.

Jordi Guàrdia has a Doctorate in Mathematics from the UB (1998) and he teaches at the **FME** and at the EPSEVG. He has also taught at the UOC (during the period 1998-2009) and at the UB (1990-2001). His research has focused on number theory, mainly in arithmetic geometry and computational algebraic number theory. He is presently working in the interaction between local number theory and singularities of curves through valuation theory. He is an active member of the Seminari de Teoria de Nombres de Barcelona and has collaborated in the organization of several international conferences and workshops, being one of the founders of the "Jornadas de Teoría de Números", which this year has reached the ninth edition.

He has served as teaching deputy of the DMAT for eleven years, and has promoted various teaching initiatives, such as the Jornada Docent of the DMAT (Department's Teacher's Day—DTD), or the coordination of teaching materials for the different engineering schools of the UPC.

In 2022 his project *Watches, dresses and roller coasters: designing with mathematics* (*Relotges, vestits i muntanyes russes: dissenyant amb matemàtiques*), developed in the context of the subject *Mathematics for Design*" (MADI for short) was distinguished with the 25th UPC Prize for Teaching Initiatives (Press release) and with the Vicens Vives Award for Teaching Quality.

The Dean's Team: 1 Jordi Guàdia, Dean; 2 Juanjo Rué Perna, Academic Secretary; 7 María Paz Linares Herreros, Vice-dean for Equality, Inclusion and Equity; 3 Sebastià Martín Molleví, Vice-dean for Promotion and External Relations; 6 Jordi Saludes Closa, Vice-dean for Teaching Innovation and New Technologies. 9 Jesús Fernández Sánchez, deputy director for the mathematics degrees; 10 Lourdes Rodero de Lamo, deputy director for the statistics degrees; 4 Gemma Flaquer, Academic Resources delegate; 8 Mireia Ribera Mitjans, Promotion and Internal Coordination delegate; 5 Jaume Fusté, Head of the FME Management.

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